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Nonpolynomial Interpolation

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Here interpolation is meant in the following sense: given $f \in C[a, b]$, and given a set of distinct points in $[a, b]$ and a linearly independent set $\{u_0,..., u_n\}$ of continuous functions on $[a, b]$, the interpolating function L_n^f is the unique linear combination of $u_0,..., u_n$ that coincides with f at the given points, if such a linear combination exists. In the classical case of Lagrange interpolation, $u_i(x)$ is a polynomial of degree i. Here we allow other choices, and prove a generalization of the mean-convergence theorem of Erdös and Turán: it is shown that if a certain condition is satisfied, then L_n^f converges to f, in an appropriate L_2 sense, for all continuous functions f for which $\mathscr{E}_n(f) \to 0$, where $\mathscr{E}_n(f)$ is the error of best uniform approximation by a linear combination of $u_0,..., u_n$. In particular, this mean-convergence property is shown to hold for interpolation by the leading eigenfunctions of a regular Sturm–Liouville eigenvalue problem, if the interpolation points are taken to be the zeros of the "next" eigenfunction. (The eigenfunctions are ordered so that the eigenvalues increase.)

1. INTRODUCTION

In this paper interpolation refers to the process of constructing a continuous curve through the points at which the values of a continuous function are known. More precisely, suppose that f is a continuous function on a closed interval $[a, b]$, and that its values are known at $n + 1$ distinct points $x_0^{(n)},..., x_n^{(n)}$ in the interval. Traditionally, the interpolation between the points has been carried out with polynomials or piecewise polynomials. Here we allow the interpolating function to be of the form

$$L_n^f(x) = \sum_{i=0}^{n} a_i u_i^{(n)},$$

where $\{u_0^{(n)},..., u_n^{(n)}\}$ is a linearly independent set of real-valued continuous functions on $[a, b]$. The coefficients $a_0,..., a_n$ are of course determined by the interpolation requirement,

$$L_n^f(x_j^{(n)}) = f(x_j^{(n)}), \qquad j = 0,..., n.$$

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Clearly, L_n^f exists and is unique for all $f \in C[a, b]$ if and only if the matrix $\{u_i^{(n)}(x_j^{(n)})\}$ is nonsingular.

An interpolating function L_n^f may exist and yet be not at all a good approximation to f. (Think of polynomial interpolation at equally spaced points.) In the present work our interest is in showing that under certain conditions L_n^f is a good approximation to f when n is large. More precisely we show that under appropriate conditions L_n^f converges to f in a certain mean-square sense; and also that the error in this sense is less than a constant multiple of the error of best uniform approximation to f by a linear combination of $u_0^{(n)}, ..., u_n^{(n)}$.

Little seems to be known about the convergence of interpolatory approximations, except for three special choices of the interpolating functions: polynomials, trigonometric polynomials, and splines. In the polynomial case it is known that the interpolation points must be carefully chosen, but that if an appropriate choice is made then the convergence behavior of L_n^f, in a variety of senses, is highly satisfactory if f is reasonably well behaved. (For summaries of known results for these cases see [1, 5–7, 10].) But for other choices of interpolating functions it seems that little or nothing is known about convergence, and consequently no guidance is available as to how to choose the interpolation points. The lack of such guidance makes interpolation with respect to nonpolynomial systems a risky enterprise in practice.

The first aim of the present work is to extend to more general systems the celebrated theorem of Erdös and Turán [2, 5, Chap. 3, Sect. 3] on the mean convergence of polynomial interpolation. The extension is stated as Theorem 1 in Section 2.

The Erdös–Turán theorem states: Let $\omega \in L_1(a, b)$ and satisfy $\omega(x) > 0$ a.e. on the finite interval $[a, b]$, and let $\{P_m\}$ be a system of polynomials orthogonal with respect to ω on the interval $[a, b]$, with P_m of degree m. If L_n^f denotes the unique polynomial of degree $\leqslant n$ that interpolates f at the zeros of $P_{n+1}(x)$, then

$$\lim_{n \to \infty} \int_a^b |L_n^f(x) - f(x)|^2 \, \omega(x) \, dx = 0, \tag{1}$$

for all $f \in C[a, b]$.

An important corollary of the Erdös–Turán theorem, via the Banach–Steinhaus theorem, is that

$$\|L_n^f - f\| \leqslant c E_n(f),$$

where c is a constant, $\|\cdot\|$ is the mean-square norm

$$\|u\| = \left[\int_a^b |u(x)|^2 \, \omega(x) \, dx \right]^{1/2}.$$

and $E_n(f)$ is the error of best uniform approximation to f by a polynomial of degree $\leqslant n$,

$$E_n(f) = \min_{p \in \mathscr{P}_n} \|f - p\|_{\mathscr{C}}.$$

Thus the Erdös–Turán theorem guarantees not only the convergence of L_n^f to f, but also its *rate* of convergence—the convergence is at least as fast as that of the best uniform approximation to f.

Theorem 1, our generalization of the Erdös–Turán theorem, requires for its statement a generalization of $E_n(f)$, namely,

$$\mathscr{E}_n(f) = \min_{c_i \in \mathscr{C},\, i=0,\ldots,n} \left\| f - \sum_{i=0}^{n} c_i u_i^{(n)} \right\|_{\mathscr{C}}.$$

Thus $\mathscr{E}_n(f)$ is the error of best uniform approximation to f by a linear combination of $u_0^{(n)},\ldots, u_n^{(n)}$. Theorem 1 states, in rough terms, that if a certain condition holds, then $\|L_n^f - f\|$ converges to zero for all continuous functions f for which $\mathscr{E}_n(f) \to 0$. Moreover, the theorem also yields

$$\|L_n^f - f\| \leqslant c\mathscr{E}_n(f),$$

for all sufficiently large n, so that as in the polynomial case the theorem guarantees not only the convergence of L_n^f to f but also its rate of convergence.

A general theorem without particular application is vacuous. We have therefore applied Theorem 1 to the case of interpolation by the eigenfunctions of a Sturm–Liouville eigenvalue problem, to obtain the following proposition. No previous convergence results for this system seem to be known. (The paper by Jensen [3] uses interpolation in a different sense from that used here.)

PROPOSITION. *Consider the eigenvalue problem*

$$p(x)\, u''(x) + q(x)\, u'(x) + [r(x) + \lambda]\, u(x) = 0, \tag{2}$$

with boundary conditions

$$\begin{aligned} \cos\alpha\ u(a) + \sin\alpha\ u'(a) &= 0, \\ \cos\beta\ u(b) + \sin\beta\ u'(b) &= 0, \end{aligned} \tag{3}$$

where $p \in C^2[a, b]$, $q \in C^1[a, b]$, $r \in C[a, b]$, $p(x) > 0$, and $q(x)$, $r(x)$, α, and β are all real. Let $u_0, u_1,\ldots,$ be the eigenfunctions, ordered so that the eigenvalues λ_n are increasing, and let L_n^f be the unique linear combination of

$u_0, \dots, u_n$ that coincides with f at the $n+1$ zeros of $u_{n+1}(x)$ that lie in the open interval (a, b). Furthermore, let

$$\omega(x) = \frac{1}{p(x)} \exp\left(\int_a^x \frac{q(x')}{p(x')}\, dx'\right). \tag{4}$$

Then the limit (1) holds for all $f \in C[a, b]$, provided f satisfies $f(a) = 0$ if $\sin\alpha = 0$, and $f(b) = 0$ if $\sin\beta = 0$. Moreover

$$\|L_n^f - f\| \leqslant c\mathscr{E}_n(f),$$

where $\mathscr{E}_n(f)$ is the error of best uniform approximation to f by linear combinations of $u_0, \dots, u_n$, and c is a constant.

The proof is given in Section 3. Because $\omega(x)$ given by (4) is bounded below by a positive constant, the result also holds if $\omega(x)$ is instead set equal to 1. However, the weight function given by (4) is in a sense the natural weight function for this problem (see Section 3).

The proposition as stated allows no freedom in the choice of interpolation points. However, for the special case $p(x) \equiv 1$, $q(x) \equiv 0$ the same result holds also for some other choices of the interpolation points. The details are given in Section 4.

To illustrate the above proposition, consider the following eigenvalue problem, based on the Bessel equation:

$$u''(x) + \frac{1}{x} u'(x) + \left(\lambda - \frac{\nu^2}{x^2}\right) u(x) = 0,$$

with boundary conditions

$$u(a) = u(b) = 0,$$

where $0 < a < b$. The eigenvalues $\lambda_n = s_n^2$ are determined by

$$J_\nu(s_n b)\, Y_\nu(s_n a) = Y_\nu(s_n b)\, J_\nu(s_n a),$$

and an eigenfunction u_n corresponding to λ_n is

$$u_n(x) = J_\nu(s_n x)\, Y_\nu(s_n a) - Y_\nu(s_n x)\, J_\nu(s_n a).$$

The proposition asserts that interpolation based on the leading eigenfunctions $u_0, \dots, u_n$, with the interpolation points taken to be the interior zeros of $u_{n+1}(x)$, converges to f in the mean-square sense for all continuous functions f that vanish at a and b. It also asserts that the mean-square error converges to zero at least as fast as the error of best uniform approximation to f by a linear combination of $u_0, \dots, u_n$.

2. GENERAL THEORY

We suppose that the interpolating functions $u_i^{(n)}$ and interpolation points $x_i^{(n)}$, $i=0,\dots,n$, are specified for all $n \geqslant 0$, that L_n^f is defined as in the first paragraph of Section 1, and that $\omega \in L_1(a,b)$ is a given nonnegative weight function. The finite-dimensional subspace spanned by $u_0^{(n)},\dots,u_n^{(n)}$ is denoted by U_n.

Corresponding to the choice of the interpolating functions $\{u_i^{(n)}\}$, there exists a natural space of continuous functions within which to set the theory: let $\mathscr{C}[a,b] \subset C[a,b]$ be the subspace of continuous functions f on $[a,b]$ for which $\mathscr{E}_n(f) \to 0$. It is easily verified that $\mathscr{C}[a,b]$ is a closed subspace of $C[a,b]$ with respect to the uniform norm, thus $\mathscr{C}[a,b]$ is a Banach space in its own right.

The first result is a necessary and sufficient condition for convergence to hold for all $f \in \mathscr{C}[a,b]$. Here $\|L_n\|$ is defined by

$$\|L_n\| = \sup_{f \in \mathscr{C}[a,b]} \frac{\|L_n^f\|}{\|f\|_\infty}.$$

LEMMA. *The limit* (1) *holds for all* $f \in \mathscr{C}[a,b]$ *if and only if*

$$\sup_n \|L_n\| < \infty. \tag{5}$$

If either condition holds then

$$\|L_n^f - f\| \leqslant c\,\mathscr{E}_n(f), \qquad f \in \mathscr{C}[a,b], \tag{6}$$

where c is a constant.

Proof. ($\Leftarrow$) Suppose that (5) holds. If $f \in \mathscr{C}[a,b]$ and $u \in U_n$ then

$$\begin{aligned}\|L_n^f - f\| &= \|L_n^{(f-u)} - (f-u)\| \\ &\leqslant \|L_n^{(f-u)}\| + \|f-u\| \\ &\leqslant \|L_n\|\,\|f-u\|_\infty + M\,\|f-u\|_\infty\end{aligned}$$

where

$$M = \left(\int_a^b \omega(x)\,dx\right)^{1/2}.$$

Since u is an arbitrary element of U_n it follows that

$$\|L_n^f - f\| \leqslant (\sup_m \|L_m\| + M)\,\mathscr{E}_n(f),$$

which converges to zero as $n \to \infty$ because $f \in \mathscr{C}[a,b]$. Also (6) clearly holds with $c = \sup_n \|L_n\| + M$.

($\Rightarrow$) It follows from the assumption that

$$\sup_n \|L_n^f\| < \infty,$$

for each f in $\mathscr{C}[a, b]$. Then (5) follows from the Banach–Steinhaus theorem, because $\mathscr{C}[a, b]$ is a Banach space. ∎

Now define the inner product

$$(u, v) = \int_a^b u(x)\, v(x)\, \omega(x)\, dx,$$

and let $\{v_0^{(n)}, \ldots, v_n^{(n)}\}$ be any basis for U_n which is orthonormal with respect to this inner product—that is,

$$(v_i, v_j) = \delta_{ij}, \qquad 0 \leqslant i, j \leqslant n.$$

(From this point on we shall omit the label n on $v_i^{(n)}$, $u_i^{(n)}$, etc. when there is no risk of confusion.) Then define

$$K_n(x, y) = \sum_{i=0}^{n} v_i(x)\, v_i(y),$$

a quantity that is clearly invariant under a change from one orthonormal basis to another, and that is closely related to orthogonal projection onto U_n: in fact $K_n(x, y)\, \omega(y)$ is the kernel of the integral operator that projects orthogonally onto U_n.

We now state the principal result. (Note that the first condition in the theorem will in most cases be satisfied only for special choices of the points $x_0, \ldots, x_n$.)

THEOREM 1. *Suppose that for all n sufficiently large we have*

$$\sum_{\substack{i=0 \\ i \neq j}}^{n} \frac{|K_n(x_i, x_j)|}{K_n(x_i, x_i)} \leqslant \rho < 1, \qquad j = 0, \ldots, n, \tag{7}$$

and

$$\sum_{i=0}^{n} \frac{1}{K_n(x_i, x_i)} \leqslant m, \tag{8}$$

where ρ and m are constants. Then L_n^f exists and is unique for n sufficiently large, and the limit (1) *holds for all $f \in \mathscr{C}[a, b]$. Moreover for n sufficiently large*

$$\|L_n^f - f\| \leqslant c\mathscr{E}_n(f),$$

where c is a constant.

The theorem is proved below. First, however, we show that for the polynomial case the theorem yields the Erdös–Turán result stated in Section 1. If we assume for convenience that the orthogonal polynomials $\{P_n\}$ satisfy $\|P_n\| = 1$, then for this case we have, from the Christoffel–Darboux identity [8, p. 43],

$$K_n(x, y) = \sum_{i=0}^{n} P_i(x) P_i(y)$$
$$= \frac{k_n}{k_{n+1}} \frac{P_{n+1}(x) P_n(y) - P_n(x) P_{n+1}(y)}{x - y},$$

where $P_n(x) = k_n x^n + \cdots$. Thus if $x_0, \ldots, x_n$ are taken to be the zeros of $P_{n+1}(x)$, then for $i \neq j$ we have $K_n(x_i, x_j) = 0$, so the condition (7) is certainly satisfied. On the other hand it is shown by Szegö [8, p. 48] that

$$K_n(x_i, x_i) = \mu_i^{-1},$$

where μ_i is the Christoffel number (or, in other words, the Gauss quadrature weight) associated with the point x_i. Thus

$$\sum_{i=0}^{n} \frac{1}{K_n(x_i, x_i)} = \sum_{i=0}^{n} \mu_i = \int_a^b \omega(x)\, dx < \infty,$$

where in the last step we have used the fact that the Gauss quadrature rule $\sum \mu_i\, g(x_i) \approx \int_a^b g(x)\, \omega(x)\, dx$ is exact for the function $g(x) \equiv 1$. Thus Theorem 1 is applicable to the polynomial case, and we recover the Erdös–Turán result.

Proof of Theorem 1. Let $\lambda_i \in U_n$ be defined by

$$\lambda_i(x) = \frac{K_n(x_i, x)}{K_n(x_i, x_i)}, \qquad i = 0, \ldots, n, \tag{9}$$

and let $\Lambda^{(n)}$ be the square matrix defined by

$$\Lambda_{ij}^{(n)} = \lambda_i(x_j) = \frac{K_n(x_i, x_j)}{K_n(x_i, x_i)}, \qquad i, j = 0, \ldots, n.$$

We shall show that $\Lambda^{(n)}$ is nonsingular if n is sufficiently large, from which it follows that L_n^f exists and is unique for large n.

If $|||\cdot|||$ denotes the matrix norm

$$|||\Lambda^{(n)}||| = \max_{0 \leqslant j \leqslant n} \sum_{i=0}^{n} |\Lambda_{ij}^{(n)}|,$$

then the first condition in the theorem is equivalent to

$$|||\Lambda^{(n)} - I^{(n)}||| \leqslant \rho < 1,$$

where $I^{(n)}$ is the unit matrix of order $n+1$. From this it follows immediately that $\Lambda^{(n)}$ is nonsingular, and in fact

$$|||\Lambda^{(n)-1}||| \leqslant (1-\rho)^{-1},$$

so that the inverses are bounded independently of n.

Now define $l_i \in U_n$ by

$$l_i = \sum_{k=0}^{n} (\Lambda^{(n)-1})_{ik} \lambda_k, \qquad i = 0, \ldots, n. \tag{10}$$

Then it is easily verified that

$$l_i(x_j) = \delta_{ij},$$

and from this it follows that the interpolating approximation L_n^f is given by

$$L_n^f(x) = \sum_{i=0}^{n} l_i(x) f(x_i). \tag{11}$$

The latter expression is analogous to that for Lagrange interpolation in the polynomial case, and the functions $\{l_0, \ldots, l_n\}$ correspond to the fundamental Lagrange polynomials.

It follows from (10) and (11) that

$$\begin{aligned} \|L_n^f\|^2 = (L_n^f, L_n^f) &= \sum_{i=0}^{n} \sum_{j=0}^{n} f(x_i)(l_i, l_j) f(x_j) \\ &\leqslant \sum_{i=0}^{n} \sum_{j=0}^{n} |(l_i, l_j)| \, \|f\|_\infty^2 \\ &\leqslant |||\Lambda^{(n)-1}|||^2 \sum_{k=0}^{n} \sum_{l=0}^{n} |(\lambda_k, \lambda_l)| \, \|f\|_\infty^2. \end{aligned}$$

Now from (9) and the definition of K_n it follows that

$$(\lambda_k, \lambda_l) = \frac{K_n(x_k, x_l)}{K_n(x_k, x_k) K_n(x_l, x_l)},$$

thus with the aid of the conditions (7) and (8) in the theorem we have

$$\sum_{k=0}^{n} \sum_{l=0}^{n} |(\lambda_k, \lambda_l)| \leqslant (1+\rho) m.$$

Therefore for n sufficiently large we have

$$\|L_n^f\|^2 \leqslant (1-\rho)^{-2}(1+\rho)\, m\, \|f\|_\infty^2,$$

and hence

$$\|L_n\| \leqslant (1-\rho)^{-1}(1+\rho)^{1/2}\, m^{1/2},$$

the right-hand side of which is independent of n. The remainder of the theorem now follows from the lemma. ∎

It may be instructive to note that in the classical case, where u_i is a polynomial of degree i and $x_0, ..., x_n$ are the zeros of $P_{n+1}(x)$, we have the simplifications

$$l_i(x) = \lambda_i(x) = \frac{K_n(x_i, x)}{K_n(x_i, x_i)} = \mu_i K_n(x_i, x),$$

$$(l_i, l_j) = (\lambda_i, \lambda_j) = \frac{K_n(x_i, x_j)}{K_n(x_i, x_i)\, K_n(x_j, x_j)} = \delta_{ij}\mu_i.$$

Thus in the classical case

$$\|L_n^f\|^2 \leqslant \sum_{i=0}^{n} \sum_{j=0}^{n} |(l_i, l_j)|\, \|f\|_\infty^2$$
$$= \sum_{i=0}^{n} \mu_i \|f\|_\infty^2 = \int_a^b \omega(x)\, dx\, \|f\|_\infty^2,$$

so that it is almost trivial to show that the norms $\|L_n\|$ are uniformly bounded. In general, however, that is not true.

3. Interpolation by Sturm–Liouville Eigenfunctions

We now prove the proposition stated in Section 1, on the mean convergence of interpolation by eigenfunctions of the boundary-value problem defined by (2) and (3).

As it stands, that boundary-value problem is not in self-adjoint form, but it becomes so on multiplying (2) by the integrating factor $\omega(x)$ defined by (4). It then follows from the classical theory that the eigenvalues are real and have their only accumulation point at $+\infty$, and that the eigenfunctions $u_0, u_1, ...$, are uniquely determined apart from a multiplicative factor, and are orthogonal with respect to the weight function $\omega(x)$.

Moreover, it follows from the work of Gantmacher and Krein (see [4, pp. 33–36]) that $u_{n+1}(x)$ has exactly $n+1$ zeros in the open interval (a, b), and also that the matrix $\{u_i(t_j)\}_{i,j=0}^n$ has nonzero determinant if $t_0,\ldots,t_n$ are any distinct points in (a, b). From these two properties it follows that the interpolating function L_n^f, which is the linear combination of $u_0,\ldots,u_n$ that interpolates f at the interior zeros of $u_{n+1}(x)$, exists and is unique for every value of n.

The task before us is to prove the mean convergence of L_n^f to f for $f \in C[a, b]$, provided that $f(a)$ vanishes if $\sin\alpha = 0$, and $f(b)$ vanishes if $\sin\beta = 0$. We first prove the result for the simpler differential equation

$$u''(x) + [r(x) + \lambda]\, u(x) = 0, \tag{12}$$

and the boundary conditions (3), and then at the end of the section extend the result to the more general problem.

The first step is to identify, for this simple problem, the space $\mathscr{C}[a, b]$, i.e., the space of continuous functions for which $\mathscr{E}_n(f) \to 0$ as $n \to \infty$. It may be shown that every continuous function f belongs to $\mathscr{C}[a, b]$ provided that $f(a) = 0$ if $\sin\alpha = 0$ and $f(b) = 0$ if $\sin\beta = 0$. Consider first the case $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. Given $f \in C[a, b]$ and $\varepsilon > 0$, the Weierstrass theorem assures us that there exists a cosine polynomial of the form

$$g(x) = \sum_{j=0}^{m} a_j \cos\left(j\pi \frac{x-a}{b-a}\right).$$

such that $\|f - g\|_\infty < \varepsilon/2$. In turn g can be uniformly approximated to an accuracy of $\varepsilon/2$ by the nth partial sum of its Sturm–Liouville series with respect to $u_0, u_1,\ldots$, if n is taken sufficiently large. This follows from a result of Titchmarsh [9, Eq. (1.9.3)], which shows that for integrable g the partial sums of the Sturm–Liouville and Fourier cosine series for g differ by a quantity that is uniformly of order $o(1)$, provided $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. (A trivial extension of Titchmarsh's argument is required to establish uniformity.) Of course with g defined as above the Fourier cosine series of g is just g itself for n sufficiently large, so the result follows. A similar argument holds if $\sin\alpha \neq 0$ but $\sin\beta = 0$, or if $\sin\alpha = \sin\beta = 0$, except that one must now use trigonometric polynomials that vanish at one or both ends as appropriate—in the latter case, for example, the cosine polynomials must be replaced by sine polynomials. The argument then goes through exactly as before, provided that $f(x)$ vanishes at one or both ends as appropriate.

The next step is to prove the mean convergence of L_n^f to f for all $f \in \mathscr{C}[a, b]$. For the present case of the differential equation (12), the weight function defined by (4) reduces to $\omega(x) \equiv 1$, so that the inner product becomes simply

$$(u, v) = \int_a^b u(x)\, v(x)\, dx.$$

If we assume for convenience that the eigenfunctions are normalized by $(u_i, u_i) \equiv \| u_i \|^2 = 1$, then the orthogonality relation for the eigenfunctions becomes

$$(u_i, u_j) = \delta_{ij}, \qquad i, j = 0, 1, \ldots.$$

Consequently, the kernel $K_n(x, y)$ can be written as

$$K_n(x, y) = \sum_{i=0}^{n} u_i(x)\, u_i(y).$$

For Theorem 1 to be applicable we must show that K_n satisfies the conditions (7) and (8), where $x_0, \ldots, x_n$ are the zeros of $u_{n+1}(x)$. Our approach to this is to develop asymptotic expressions for $K_n(x_i, x_j)$ by means of the contour integral methods used by Titchmarsh [9, Chap. 1] to study Sturm–Liouville series.

If λ is an arbitrary complex number, Titchmarsh shows that Green's function for the differential operator in (12) and the boundary conditions (3) is

$$\begin{aligned} G(x, y; \lambda) &= \chi(x; \lambda)\, \phi(y; \lambda)/\omega(\lambda), \qquad y \leqslant x, \\ &= \phi(x; \lambda)\, \chi(y; \lambda)/\omega(\lambda), \qquad x \leqslant y, \end{aligned}$$

where ϕ satisfies the differential equation (12) and the boundary conditions

$$\phi(a; \lambda) = \sin \alpha, \qquad \phi'(a; \lambda) = -\cos \alpha, \tag{13}$$

χ satisfies (12) and the boundary conditions

$$\chi(b; \lambda) = \sin \beta, \qquad \chi'(b; \lambda) = -\cos \beta, \tag{14}$$

and $\omega(\lambda)$ is the Wronskian

$$\omega(\lambda) = \phi(x; \lambda)\, \chi'(x; \lambda) - \phi'(x; \lambda)\, \chi(x; \lambda).$$

Titchmarsh also shows that $\omega(\lambda)$ is an entire function of λ with zeros at the eigenvalues $\lambda_0, \lambda_1, \ldots$, and that the residue of $G(x, y; \lambda)$ at $\lambda = \lambda_n$ is $u_n(x)\, u_n(y)$. It follows that $K_n(x, y)$ can be evaluated by integrating $G(x, y; \lambda)$ around an appropriate contour in the λ plane.

Now let $\lambda = s^2$, and write $s = \sigma + it$, with $\sigma \geqslant 0$. Then Titchmarsh shows [9, p. 10] that if $|s|$ is sufficiently large, then

$$\phi(x; \lambda) = \sin \alpha \cos s(x - a) + O(|s|^{-1} e^{|t|(x-a)}),$$

if $\sin \alpha \neq 0$, and

$$\phi(x; \lambda) = -\cos \alpha \frac{\sin s(x - a)}{s} + O(|s|^{-2} e^{|t|(x-a)}),$$

if $\sin\alpha = 0$. Similarly,

$$\phi'(x;\lambda) = -s \sin\alpha \sin s(x-a) + O(e^{|t|(x-a)}),$$

if $\sin\alpha \neq 0$, and

$$\phi'(x;\lambda) = -\cos\alpha \cos s(x-a) + O(|s|^{-1} e^{|t|(x-a)}),$$

if $\sin\alpha = 0$. Similar asymptotic expressions hold for $\chi(x;\lambda)$ and $\chi'(x;\lambda)$.

In each of the above expressions, and in similar expressions throughout this section, the error terms are *uniform* for $a \leqslant x \leqslant b$. Thus, for example,

$$|O(|s|^{-1} e^{|t|(x-a)})| \leqslant c\,|s|^{-1} e^{|t|(x-a)},$$

where c is independent of x and s.

It follows from the asymptotic expressions for $\phi(x;\lambda)$ and $\chi(x;\lambda)$ that

$$\omega(\lambda) = \sin\alpha \sin\beta\, s \sin s(b-a) + O(e^{|t|(b-a)}), \tag{15}$$

if $\sin\alpha \neq 0$ and $\sin\beta \neq 0$,

$$\omega(\lambda) = -\sin\alpha \cos\beta \cos s(b-a) + O(|s|^{-1} e^{|t|(b-a)}),$$

if $\sin\alpha \neq 0$ and $\sin\beta = 0$, and

$$\omega(\lambda) = \cos\alpha \cos\beta \frac{\sin s(b-a)}{s} + O(|s|^{-2} e^{|t|(b-a)}),$$

if $\sin\alpha = \sin\beta = 0$.

First consider the case $\sin\alpha \neq 0$, $\sin\beta \neq 0$. Following Titchmarsh [9 p. 13], for this case we take the upper half of the λ contour to correspond to the quarter square in the s plane defined by

$$\sigma = \frac{(n+(1/2))\,\pi}{b-a}, \qquad 0 \leqslant t \leqslant \frac{(n+(1/2))\,\pi}{b-a}, \tag{16}$$

and

$$\frac{(n+(1/2))\,\pi}{b-a} \geqslant \sigma \geqslant 0, \qquad t = \frac{(n+(1/2))\,\pi}{b-a}. \tag{17}$$

The lower half of the λ contour is then obtained by making the contour symmetric about the real axis.

On this contour we have

$$|\sin s(b-a)| > A e^{|t|(b-a)},$$

where A is a positive constant. Hence it follows from (15) and from Rouché's theorem [9, p. 19] that for n sufficiently large there are exactly

$n+1$ zeros of $\omega(\lambda)$ inside the contour. Moreover, for λ on the above contour we have

$$\frac{1}{\omega(\lambda)} = \frac{1}{\sin\alpha \sin\beta\, s \sin s(b-a)} \left[1 + O\left(\frac{1}{|s|}\right)\right].$$

Thus for $\sin\alpha \neq 0$ and $\sin\beta \neq 0$ it follows that

$$G(x, y; \lambda) = \frac{\cos s(b-x) \cos s(y-a)}{s \sin s(b-a)} + O\left(\frac{1}{|s|^2} e^{-|t|(x-y)}\right), \tag{18}$$

for λ on the contour and $y \leqslant x$.

A similar argument holds if $\sin\alpha \neq 0$ and $\sin\beta = 0$, except that the contour needs to be modified by replacing $n+\frac{1}{2}$ by $n+1$ in (16) and (17), and $\sin s(b-a)$ has to be replaced by $\cos s(b-a)$. The resulting asymptotic expression for $G(x, y; \lambda)$ is

$$G(x, y; \lambda) = -\frac{\sin s(b-x) \cos s(y-a)}{s \cos s(b-a)} + O\left(\frac{1}{|s|^2} e^{-|t|(x-y)}\right).$$

for λ on the contour and $y \leqslant x$. Finally, if $\sin\alpha = \sin\beta = 0$, then $n+\frac{1}{2}$ needs to be replaced by $n+\frac{3}{2}$ in (16) and (17), and the result is

$$G(x, y; \lambda) = -\frac{\sin s(b-x) \sin s(y-a)}{s \sin s(b-a)} + O\left(\frac{1}{|s|^2} e^{-|t|(x-y)}\right),$$

for λ on the contour and $y \leqslant x$.

In each of the three cases we have, for n sufficiently large,

$$K_n(x, y) = \frac{1}{2\pi i} \int G(x, y; \lambda)\, d\lambda,$$

if the integration is taken around the appropriate λ contour defined above. For the case $\sin\alpha \neq 0$ and $\sin\beta \neq 0$ we find, by using (18) and then integrating explicitly,

$$\begin{aligned} K_n(x, y) &= \frac{1}{b-a}\left[D_n\left(\pi\frac{x-y}{b-a}\right) + D_n\left(\pi\frac{x+y-2a}{b-a}\right)\right] \\ &\quad + O\left(\frac{1}{n|x-y|}\right), \qquad x \neq y, \\ K_n(x, x) &= \frac{1}{b-a}\left[n + D_n\left(\pi\frac{2x-2a}{b-a}\right)\right] + O(1), \end{aligned} \tag{19}$$

where $D_n(\theta)$ is the Dirichlet kernel,

$$D_n(\theta) = \frac{1}{2} + \sum_{i=1}^{n} \cos i\theta = \frac{\sin(n + (1/2))\,\theta}{2 \sin \theta/2}.$$

Similarly, for the case $\sin\alpha \neq 0$ and $\sin\beta = 0$ we find

$$\begin{aligned} K_n(x, y) &= \frac{1}{b-a} \left[D_n^{(1)}\left(\pi \frac{x-y}{b-a}\right) + D_n^{(1)}\left(\pi \frac{x+y-2a}{b-a}\right) \right] \\ &\quad + O\left(\frac{1}{n\,|x-y|}\right), \qquad x \neq y, \\ K_n(x, x) &= \frac{1}{b-a} \left[n + D_n^{(1)}\left(\pi \frac{2x-2a}{b-a}\right) \right] + O(1), \end{aligned} \tag{20}$$

where

$$D_n^{(1)}(\theta) = \sum_{i=0}^{n} \cos\left(i + \frac{1}{2}\right)\theta = \frac{\sin(n+1)\,\theta}{2 \sin \theta/2}.$$

and for the case $\sin\alpha = \sin\beta = 0$

$$\begin{aligned} K_n(x, y) &= \frac{1}{b-a} \left[D_{n+1}\left(\pi \frac{x-y}{b-a}\right) - D_{n+1}\left(\pi \frac{x+y-2a}{b-a}\right) \right] \\ &\quad + O\left(\frac{1}{n\,|x-y|}\right), \qquad x \neq y, \\ K_n(x, x) &= \frac{1}{b-a} \left[n - D_{n+1}\left(\pi \frac{2x-2a}{b-a}\right) \right] + O(1). \end{aligned} \tag{21}$$

The next step is to find asymptotic estimates for the zeros of $u_{n+1}(x)$. That requires first the development of asymptotic estimates for the eigenvalues $\lambda_n = s_n^2$, $n = 0, 1,\ldots$. Consider the case $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. For this case we know already that

$$\frac{(n-(1/2))\,\pi}{b-a} < s_n < \frac{(n+(1/2))\,\pi}{b-a},$$

if n is sufficiently large. A tighter estimate for s_n follows from the property $\omega(\lambda_n) = 0$, which by (15) implies

$$s_n \sin s_n(b-a) = O(1).$$

On substituting $s_n = n\pi/(b-a) + \tau_n$ it follows readily that $\tau_n = O(1/n)$, and hence

$$s_n = \frac{n\pi}{b-a} + O\left(\frac{1}{n}\right).$$

Similarly, for the case $\sin\alpha \neq 0$ and $\sin\beta = 0$

$$s_n = \frac{(n+(1/2))\,\pi}{b-a} + O\left(\frac{1}{n}\right),$$

and for the case $\sin\alpha = \sin\beta = 0$

$$s_n = \frac{(n+1)\,\pi}{b-a} + O\left(\frac{1}{n}\right).$$

From the classical results on the zeros of eigenfunctions [9, pp. 107–108], it follows that the interior zeros of $u_n(x)$ lie between the corresponding zeros of $\phi_n^{\pm}(x)$, where $\phi_n^{\pm}$ are the solutions of

$$\phi_n^{\pm\,''} + (R^{\pm} + \lambda_n)\,\phi_n^{\pm} = 0,$$
$$\phi_n^{\pm}(a) = \sin\alpha, \qquad \phi_n^{\pm\,'}(a) = -\cos\alpha,$$

and where

$$R^{+} = \max_{a \leqslant x \leqslant b} r(x), \qquad R^{-} = \min_{a \leqslant x \leqslant b} r(x).$$

For the case $\sin\alpha \neq 0$ and $\sin\beta \neq 0$

$$\phi_n^{\pm}(x) = c\cos(s_n^{\pm}(x-a) + \delta_n^{\pm}),$$

where c is a constant,

$$s_n^{\pm} = (R^{\pm} + \lambda_n)^{1/2} = \frac{n\pi}{b-a}\,[1 + O(1/n^2)],$$

and, from the boundary conditions,

$$\delta_n^{\pm} = O(1/n).$$

Thus the zeros of $\phi_n^{\pm}(x)$, and hence also of $u_n(x)$, are given by $a + [(i+\frac{1}{2})/n](b-a) + O(1/n^2)$, $i = 0,\dots, n-1$. On replacing n by $n+1$ we obtain for our interpolation points

$$x_i^{(n)} = a + \frac{i+(1/2)}{n+1}\,(b-a) + O\left(\frac{1}{n^2}\right), \qquad i = 0,\dots, n. \tag{22}$$

Similarly, for the case $\sin\alpha \neq 0$ and $\sin\beta = 0$

$$x_i^{(n)} = a + \frac{i + (1/2)}{n + (3/2)}(b - a) + O\left(\frac{1}{n^2}\right), \qquad i = 0, \ldots, n, \tag{23}$$

and for the case $\sin\alpha = \sin\beta = 0$

$$x_i^{(n)} = a + \frac{i + 1}{n + 2}(b - a) + O\left(\frac{1}{n^2}\right), \qquad i = 0, \ldots, n \tag{24}$$

We now seek to show that for these interpolation points the conditions (7) and (8) are satisfied. In rough terms this is possible only because the off-diagonal elements of $K_n(x_i, x_j)$ are very small for these points, while the diagonal elements are large. (That is certainly not true for arbitrary choices of the points.) We shall work out in detail the case $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, for which $K_n(x, y)$ is given by (19), and the interpolation points by (22). With the aid of the easily derived formula

$$D_n(\theta) = \frac{\sin(n + 1)\theta}{2\tan\theta/2} - \frac{1}{2}\cos(n + 1)\theta,$$

we find for this case

$$D_n\left(\pi\frac{x_i - x_j}{b - a}\right) = -\frac{1}{2}(-1)^{i-j} + O\left(\frac{1}{|i - j|}\right), \qquad i \neq j,$$

and

$$D_n\left(\pi\frac{x_i + x_j - 2a}{b - a}\right) = \frac{1}{2}(-1)^{i+j} + O\left(\frac{1}{|i - j|}\right), \qquad i \neq j,$$

$$= O(1), \qquad i = j.$$

Thus from (19) we have

$$K_n(x_i, x_j) = O\left(\frac{1}{|i - j|}\right), \qquad i \neq j \tag{25}$$

(because of a cancellation of the $O(1)$ terms), and

$$K_n(x_i, x_i) = \frac{n}{b - a} + O(1). \tag{26}$$

Thus

$$\max_{0 \leqslant j \leqslant n} \sum_{\substack{i=0 \\ i \neq j}}^{n} \frac{|K_n(x_i, x_j)|}{K_n(x_j, x_j)} = O\left(\frac{\log n}{n}\right),$$

and

$$\sum_{i=0}^{n} \frac{1}{K_n(x_i, x_i)} = O(1).$$

Therefore the conditions of Theorem 1 are satisfied if n is sufficiently large.

A similar argument yields the estimates (25) and (26) again if $\sin \alpha \neq 0$ and $\sin \beta = 0$, or if $\sin \alpha = \sin \beta = 0$. Thus in all three cases the conditions of Theorem 1 are satisfied for large n. It then follows from Theorem 1 that the proposition stated in Section 1 is valid for the particular boundary-value problem defined by (12) and (3).

The last step is to extend the results to the more general boundary-value problem defined by (2) and (3). Now Titchmarsh [9, p. 22] points out that the transformations

$$t = \int_a^x [p(x')]^{-1/2}\, dx', \tag{27}$$

and

$$u(x) = h(x)\, w(t), \tag{28}$$

where

$$h(x) = p(x)^{1/4} \exp\left(-\frac{1}{2}\int_a^x \frac{q(x')}{p(x')}\, dx'\right),$$

transform the differential equation (2) into the equation

$$w''(t) + [\gamma(t) + \lambda]\, w(t) = 0, \tag{29}$$

where

$$\gamma(t) = \frac{1}{4} p''(x) + \frac{p'(x)q(x)}{2p(x)} - \frac{3p'(x)^2}{16p(x)} - \frac{1}{2} q'(x) - \frac{q(x)^2}{4p(x)} + r(x).$$

The new interval is $[0, d]$, where

$$d = \int_a^b [p(x)]^{-1/2}\, dx.$$

Obviously $w(t)$ vanishes at 0 or d if and only if $u(x)$ vanishes at a or b, respectively. More generally, the boundary conditions (3) transform into analogous boundary conditions for $w(t)$, which we write as

$$\begin{aligned} \cos \xi\, w(0) + \sin \xi\, w'(0) &= 0, \\ \cos \eta\, w(d) + \sin \eta\, w'(d) &= 0, \end{aligned} \tag{30}$$

where ξ and η are real numbers.

Let the eigenvalues of the boundary-value problem defined by (29) and (30) be $\lambda_0, \lambda_1, \ldots$ and let the corresponding eigenfunctions be $w_0, w_1, \ldots$. Then the original boundary-value problem has the same eigenvalues, and the eigenfunctions $u_0, u_1, \ldots$ given by

$$u_n(x) = h(x)\, w_n(t).$$

A simple argument now establishes for the general problem that $\mathscr{C}[a, b]$, i.e., the space of continuous functions that can be uniformly approximated by linear combinations of $u_0, \ldots, u_n$ in the sense that $\mathscr{E}_n(f) \to 0$, is just $C[a, b]$ if $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, and is the set of continuous functions that vanish at a or b, respectively, if $\sin\alpha = 0$ or $\sin\beta = 0$. We argue as follows: To each $f \in C[a, b]$ we may define a corresponding function $\tilde{f} \in C[0, d]$ by

$$f(x) = h(x)\tilde{f}(t).$$

If h^- and h^+ denote the minimum and maximum values of $|h(x)|$, then it is easily seen that

$$h^- \left\| \tilde{f} - \sum_{i=0}^{n} c_i w_i \right\|_\infty \leqslant \left\| f - \sum_{i=0}^{n} c_i u_i \right\|_\infty \leqslant h^+ \left\| \tilde{f} - \sum_{i=0}^{n} c_i w_i \right\|_\infty$$

if $c_0, \ldots, c_n$ are any real numbers. It follows that f can be uniformly approximated on $[a, b]$ by a linear combination of $u_0, \ldots, u_n$ if and only if $\tilde{f}$ can be uniformly approximated on $[0, d]$ by a linear combination of $w_0, \ldots, w_n$. But we know already that the latter holds provided that $\tilde{f}(0) = 0$ if $\sin\xi = 0$ (or, equivalently, if $\sin\alpha = 0$) and $\tilde{f}(d) = 0$ if $\sin\eta = 0$ (or, equivalently, if $\sin\beta = 0$). It follows that $f \in \mathscr{C}[a, b]$ provided only that $f(a) = 0$ if $\sin\alpha = 0$, and $f(b) = 0$ if $\sin\beta = 0$.

Finally, we establish the mean-convergence property of L_n^f. With $f \in \mathscr{C}[a, b]$ and $\tilde{f}$ defined as above, let $\mathscr{L}_n^{\tilde{f}}$ denote the unique linear combination of $w_0, \ldots, w_n$ that interpolates $\tilde{f}$ at the interior zeros of $w_{n+1}(t)$. Then it is easily seen that

$$L_n^f(x) = h(x)\, \mathscr{L}_n^{\tilde{f}}(t),$$

and in consequence,

$$\int_a^b |L_n^f(x) - f(x)|^2\, h(x)^{-2}\, p(x)^{-1/2}\, dx = \int_0^d |\mathscr{L}_n^{\tilde{f}}(t) - \tilde{f}(t)|^2\, dt.$$

But the right-hand side converges to zero by the limited form of the proposition established for the special case earlier in this section. Therefore we conclude that

$$\lim_{n \to \infty} \int_a^b |L_n^f(x) - f(x)|^2\, \omega(x)\, dx = 0,$$

where

$$\omega(x) = h(x)^{-2} p(x)^{-1/2} = \frac{1}{p(x)} \exp\left(\int_a^x \frac{q(x')}{p(x')}\, dx'\right).$$

In a similar way we also obtain

$$\begin{aligned} \|L_n^f - f\| &= \left[\int_a^b |L_n^f(x) - f(x)|^2 \,\omega(x)\, dx\right]^{1/2} \\ &= \left[\int_0^d |\mathscr{L}_n^{\tilde{f}}(t) - \tilde{f}(t)|^2 \, dt\right]^{1/2} \\ &\leqslant c \min_{c_0,\dots,c_n} \left\| \tilde{f} - \sum_{i=0}^n c_i w_i \right\|_\infty \\ &\leqslant c' \min_{c_0,\dots,c_n} \left\| f - \sum_{i=0}^n c_i u_i \right\|_\infty = c' \mathscr{E}_n(f), \end{aligned}$$

where c' is a constant. The proof of the proposition stated in Section 1 is now complete. ∎

4. Further Results

For the particular eigenvalue problem defined by (12) and (3), the conditions in Theorem 1 are also satisfied for some other choices of interpolation points. The known results are stated in the following:

LEMMA. *Let the interpolating functions be the eigenfunctions $u_0,\dots, u_n$ of the boundary-value problem defined by* (12) *and* (3), *with the eigenfunctions ordered so that the eigenvalues increase. Then the limit* (1) *holds, with $\omega(x) \equiv 1$, if the interpolation points x_i, $i = 0,\dots, n$, are given by any of*

(a) *if* $\sin\alpha \neq 0$ *and* $\sin\beta \neq 0$

$$x_i = a + [(i + \tfrac{1}{2})/(n+1)](b-a),$$

or

$$x_i = a + [i/(n+\tfrac{1}{2})](b-a),$$

or

$$x_i = a + [(i+\tfrac{1}{2})/(n+\tfrac{1}{2})](b-a);$$

(b) *if* $\sin\alpha \neq 0$ *and* $\sin\beta = 0$

$$x_i = a + [(i + \tfrac{1}{2})/(n + \tfrac{1}{2})](b - a),$$

or

$$x_i = a + [i/(n + 1)](b - a),$$

or

$$x_i = a + [(i + \tfrac{1}{2})/(n + 1)](b - a);$$

(c) *if* $\sin\alpha = \sin\beta = 0$

$$x_i = a + [(i + 1)/(n + 2)](b - a),$$

or

$$x_i = a + [(i + \tfrac{1}{2})/(n + \tfrac{3}{2})](b - a),$$

or

$$x_i = a + [(i + 1)/(n + \tfrac{3}{2})](b - a).$$

The first formula for x_i in each of the three cases is just the corresponding asymptotic formula for the interior zeros of $u_{n+1}(x)$ (see (22)–(24)), hence for these points the result has effectively been established already. The other results follow in a similar way, starting from the asymptotic estimates (19)–(21) for $K_n(x, y)$.

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